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A Modified Simplex Method for Solving Linear-Quadratic and Linear Fractional Bi-Level Programming Problem

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ABSTRACT
The bi-level programming problem (BLP) is a suitable method for solving the real and complex problems in applicable areas such as management, economics, policies and planning and so on. There are several forms of the BLP as an NP-hard problem. The linear-quadratic bi-level programming (LQBP) and the linear-fractional bi-level programming (LFBP) are important forms of the BLP. In this paper, we attempt to develop two effective approaches, one based on modified simplex method and the other based on the genetic algorithm for solving the LQBP and LFBP. To obtain efficient upper bound and lower bound we employ the Karush -Kuhn -Tucker (KKT) conditions for transforming the LQBP into a single level problem. By using the proposed penalty functions, the single problem is transformed to an unconstraint problem and then it is solved by modified simplex method and genetic algorithm. The proposed approach achieves efficient and feasible solution and it is evaluated by comparing with references.

Keyword: The linear-quadratic bi-level programming, the penalty function method, simplex method, Karush-Kuhn–Tucker conditions.

1. INTRODUCTION
It has been proved that the BLP is NP- Hard problem even to seek for the locally optimal solutions (Bard ,1991; Vicente, et al., 1994)[3, 23]. Nonetheless the BLPP is an applicable problem and practical tool to solve decision making problems. It is used in several areas such as transportation, finance and so on. Therefore finding the optimal solution has a special importance to researchers. Several algorithms have been presented for solving the BLP (Yibing, et al., 2007; Allende & G. Still, 2012; Mathieu, et al., 1994; Wang, et al., 2008; Wend & U. P. Wen, 2000; Bard, 1998, Facchinei, et al.,)[30, 1, 17, 25, 24, 4, 6]. These algorithms are divided into the following classes: Transformation methods (Luce, et al., 2013; Dempe & Zemkoho, 2012) [15, 5], Fuzzy methods (Sakava et al., 1997; Sinha 2003; Pramanik & T.K. Ro 2009; Arora & Gupta 2007; Masatoshi & Takeshi.M 2012; Zhongping & Guangmin.W 2008, Zheng, et al., 2014) [20, 21, 19, 2, 16, 32, 33], Global techniques (Nocedal & S.J. Wright, 2005; Khayyal, 1985; Mathieu, et al., 1994; Wang et

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al., 2008, Wan, et al., 2014, Xu, et al., 2014, Hosseini, E and I.Nakhai Kamalabadi., 2014,) [18, 13, 17, 25, 27, 28, 10, 34], Primal–dual interior methods (Wend & U. P. Wen, 2000) [24], Enumeration methods (Thoai, et al., 2002) [22], Meta heuristic approaches (Hejazi, et al., 2002; Wang et al., 2008; Hu, et al., 2010; Baran Pal, et al., 2010; Wan et al., 2012; Yan, et al., 2013; Kuen-Ming et al., 2007, Hosseini, E and I.Nakhai Kamalabadi., 2013, He, X and C. Li, T. Huang, 2014) [11, 25, 12, 4, 26, 29, 14, 8, 9, 7]. In the following, these techniques are shortly introduced.

1.1. **Transformation methods**

An important class of methods for constrained optimization seeks the solution by replacing the original constrained problem with a sequence of unconstrained sub-problems or a problem with simple constraints. These methods are interested by some researchers for solving BLPP, so that they transform the follower problem by methods such as penalty functions, barrier functions, Lagrangian relaxation method or KKT conditions. In fact, these techniques convert the BLPP into a single problem and then it is solved by other methods [3, 4, 22, 23, 32].

1.2. **Meta heuristic approaches**

Meta heuristic approaches are proposed by many researchers to solve complex combinatorial optimization. Whereas these methods are too fast and known as suitable techniques for solving optimization problems, however, they can only propose a solution near to optimal. These approaches are generally appropriate to search global optimal solutions in very large space whenever convex or non-convex feasible domain is allowed. In these approaches, BLPP is transformed to a single level problem by using transformation methods and then meta heuristic methods are utilized to find out the optimal solution [15, 16, 17, 18, 19, 25, 33].

The remainder of the paper is structured as follows: in Section 2, basic concepts of the linear quadratic and linear fractional are introduced. The first presented algorithm is proposed in Section 3. In Section 4 and computational results are presented for approach in Section 5. Finally, the paper is finished in Section 6 by presenting the concluding remarks.

2. The concepts of the problems

We research two special classes of bi-level programming: linear-quadratic bi-level programming (LQBP) and Linear-fractional bi-level programming (LFBP). The LQBP is formulated as follows [16]:

$$\begin{aligned}
&\max_x \quad f(x,y)=a^T x+b^T y \\
&\quad s.t \ \max_y \quad g(x,y)=c^T x+d^T y+(x^T,y^T)Q(x^T,y^T)^T \\
&\qquad\quad s.t \quad Ax+By\le r, \\
&\qquad\qquad\quad x,y\ge 0.
\end{aligned} \tag{1}$$

Where $a\,,c\in R^{n_1},b\,,d\in R^{n_2},A\in R^{m\times n_1},B\in R^{m\times n_2}$, $r\in R^m$ $,x\in R^{n_1}$, $y\in R^{n_2}$ and f (x, y), g (x, y) are the objective functions of the leader and the follower, respectively. Also $Q\in R^{n_1+n_2}\times R^{n_1+n_2}$ is symmetric positive semi –definite matrix.

Suppose that

$$Q=\begin{bmatrix} Q_2 & Q_1^T \\ Q_1 & Q_0 \end{bmatrix}$$

Which $Q_0\in R^{n_2\times n_2}$, $Q_1\in R^{n_2\times n_1}$, $Q_2\in R^{n_1\times n_1}$.

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Then the follower problem of the LQBP is

$$\max_{y} \quad g(x,y) = d^T y + 2Q_1 xy + y^T Q_0 y \qquad (2)$$
$$s.t \quad By \le r - Ax,$$
$$y \ge 0.$$

The LFBP problem is formulated as follows [21]:

$$\max_{x} \quad f(x,y) = \frac{a_1 + a_2^T x + a_3^T y}{b_1 + b_2^T x + b_3^T y}$$
$$s.t \ \max_{y} \quad g(x,y) = c^T x + d^T y \qquad (3)$$
$$s.t \quad Ax + By \le r,$$
$$x, y \ge 0.$$

Which a_1 , $b_1 \in R$, a_2 , $b_2, c \in R^{n_1}$, a_3, b_3 , $d \in R^{n_2}$, $A \in R^{m \times n_1}$, $B \in R^{m \times n_2}$, $r \in R^m$, $x \in R^{n_1}$, $y \in R^{n_2}$.

The feasible region of the LQBP and LFBP problems is

$$S = \{(x,y) \mid Ax + By \le r \ , \ x, y \ge 0\}. \qquad (4)$$

On the other hand if x be fixed, the feasible region of the follower can be explained as

$$S(x) = \{y \mid By \le r - Ax \ , \ x, y \ge 0\}. \qquad (5)$$

Based on the above assumptions the follower rational reaction set is

$$P(x) = \{y \mid y \in \arg\max[g(x,y) \mid y \in S(x)\}. \qquad (6)$$

Where the inducible region is as follows

$$IR = \{(x,y) \in S, y \in P(x)\}. \qquad (7)$$

Finally the bi-level programming problem can be written as

$$\max\{f(x,y) \mid (x,y) \in IR\}. \qquad (8)$$

If there is finite solution for the BLP problem, we define feasibility and optimality for the BLP problem as

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$$S = \{(x, y) \mid Ax + By \le r, x, y \ge 0\}. \qquad (9)$$

Definition 1:

(x, y) is a feasible solution to bi-level problem if $(x, y) \in IR.$

Definition 2:

(x^*, y^*) is an optimal solution to the problem if

$$f(x^*, y^*) \le f(x, y) \quad \forall (x, y) \in IR. \qquad (10)$$

3. Modified simplex algorithm by penalty function method for LQBP and LFBP

Penalty functions transform a constrained problem into a single unconstrained problem or into a sequence of unconstrained problems. In this method the constraints are replaced into the objective function via a penalty parameter in a way that penalizes any violation of the constraints. In general, a suitable function must incur a positive penalty for infeasible points and no penalty for feasible points. Also the penalty function method is a common approach to solve the bi-level programming problems. In this kind of approach the lower level problem is appended to the upper level objective function with a penalty.

Since in problem (2), most of the equality constraints are not linear then it concerns that the above problem is non-convex programming, which indicates there are local optimal solutions that they are not global solution. Therefore solving the problem (2) is complicated and we use the following method for solving this problem.We use a penalty function to convert problem (2) to an unconstraint problem. Consider the problem (2), we append all constraints to the upper level objective function with a penalty for each constraint, then we obtain the following penalized problem.

$$\begin{aligned} &\max\ a^T x + b^T y - M(2Q_1 x + 2Q_0 y - Bu + v + d) - N(uw + vy) \\ &\quad s.t \\ &\qquad Ax + By + w = r \qquad (11) \\ &\qquad x, y, u, v, w \ge 0. \end{aligned}$$

Which N is a large positive number and M is a matrix of large positive numbers and $M \in R^{n_1+n_2}$.

We now show that two problems (2), (11) have a same optimal solution according to the following theorem, and then solve the problem (11) instead problem (2) using proposed modified simplex method. Above method is satisfied for problem (3) too.

We now propose a theorem which establishes the convergence of algorithms for solving a problem of the form: minimize $f(x)$ subject to $x \in R^n$. We show that an algorithm that generates n linearly independent search directions, and obtains a new point by sequentially minimizing f along these directions, converges to a stationary point. The theorem also establishes the convergence of algorithms using linearly independent and orthogonal search directions.
same optimal solution according to the following theorem.

Theorem 3.1:
Consider the following problem:

$$\begin{aligned} &\min_x f(x) \\ &s.t\ g_i(x) \le 0,\ i=1,2,\ldots,m, \end{aligned} \qquad (12)$$

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$h_j\ (x) = 0,$ j=1,2,…,l,

where $f, g_1, \ldots, g_m,\ h_1, \ldots, h_l$ are continuous functions on R^n and X is a nonempty set in R^n. Suppose that the problem

has a feasible solution, and α is a continuous function as follows:

$$\alpha(\mathrm{x}) = \sum_{i=1}^{m} \emptyset[g_i(x)] + \sum_{i=1}^{l} \emptyset[h_i(x)] \qquad (13)$$

where

$$\emptyset(y) = 0 \text{ if } \mathrm{y} \le 0,\ \emptyset(y) > 0 \ if\ \ y > 0. \qquad (14)$$

$$\emptyset(y) = 0 \text{ if } \mathrm{y} = 0,\ \emptyset(y) > 0 \ if\ \ y \ne 0. \qquad (15)$$

Then,

$$\begin{aligned} \inf\{f(x) : g(x) \le 0,\ \ h(x) = 0, x \in X\} \\ = \inf\{f(x) + ⍰\alpha(x) : x \in X\} \end{aligned} \qquad (16)$$

where ⍰ is a large positive constant (⍰ → ∞).

Proof:

Let y be a feasible point and $\varepsilon > 0$. Let x_1 be an optimal solution to the problem to minimize $f(x) + \mu\alpha(x)$ subject to $x \in X,\ \ \mu = 1.$ If $\mu \ge \frac{1}{\varepsilon}\left|f(y) - f(x_1)\right| + 2$, because $f(x_\mu), \theta(\mu), \alpha(x_\mu)$ are non-increasing functions, then, we must have $f(x_\mu) \ge f(x_1).$

Now we show that $\alpha(x_\mu) > \varepsilon,$ By contradiction, suppose that, $\alpha(x_\mu) \le \varepsilon.$ then

$$\begin{aligned} \inf\{f(x) : g(x) \le 0,\ h(x) = 0\} \ge \theta(\mu) = f(x_\mu) + \mu\alpha(x_\mu) \ge f(x_1) + \mu\alpha(x_\mu) \\ > f(x_1) + \left|f(y) - f(x_1)\right| + 2\varepsilon > f(y). \end{aligned}$$

The above inequality is not possible in view of feasibility of y. thus, $\alpha(x_\mu) \le \varepsilon$ for all

$\mu \ge \frac{1}{\varepsilon}\left|f(y) - f(x_1)\right| + 2.$ Since $\varepsilon > 0$ is arbitrary, $\alpha(x_\mu) \to 0$ as $\mu \to \infty.$

Since $\mu \to \infty,$ as $\alpha(\bar{x}) = 0,$ then $\alpha(x_\mu) \to 0$ that is, $\bar{x}$ is a optimal solution to the original problem and that $\sup \theta(\mu) = f(\bar{x})$ note that $\mu\alpha(x_\mu) = \theta(\mu) - f(x_\mu).$ As $\mu \to \infty, \theta(\mu)\, and\ f(x_\mu)$ Both approach $f(\bar{x})$ and hence, $\mu\alpha(x_\mu)$ approaches zero. This completes the proof.

According to the above theorem two problems (2), (11) have a same optimal solution. The modified simplex method is proposed as follows.
Steps of the modified simplex algorithm are proposed as follows:

Let the main iteration number, k=0 and the objective function value at the optimal solution at the k-th iteration $Z_0^* = -\infty.$

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Step 1:

If $Ax + By + w = r$ in the problem (11) is infeasible go to Step 5. Otherwise find an arbitrary basic feasible solution of $Ax + By + w = r$. Let (x_B, y_B, w_B) be the associated basic inverses H^{-1}. The variables are divided into two separate classes, basic and non-basic variables that basic variables can be written according to the non-basic variables as follows:

$$\begin{bmatrix} x_B \\ y_B \\ w_B \end{bmatrix} = H^{-1}(b - P_N x_N - Q_N y_N - I_N w_N)$$

Where P_N , Q_N , I_N are matrixes which correspond with the columns of the non-basic variables

x_N , y_N , w_N respectively.

Step 2:

By replacing equation $\begin{bmatrix} x_B \\ y_B \\ w_B \end{bmatrix} = H^{-1}(b - P_N x_N - Q_N y_N - I_N w_N)$ into the objective function of the problem (11), this objective function can be written as follows:

$$z = z_0 + a^T x + b^T y - M(2Q_1 x + 2Q_0 y - Bu + v + d) - N(uw + vy)$$

Which z_0 is the current value of the objective function. For all non-basic variables we calculate $z_j - c_j$.
According to the usual rule in the simplex method if all $z_j - c_j$ are positive, the simplex method will be finished and then we go to Step 4. Otherwise go to Step 3.

Step 3:

According to the usual rule in the simplex method enter the non-basic primal variable with the smallest $z_j - c_j$ or the non-basic dual variable with the largest $z_j - c_j$ into the basis. Also the leaving variable is determined using the usual minimum ratio rule:

$$\frac{r_k}{y_{kj}} = \min\{\frac{r_i}{y_{ij}} \mid y_{ij} \geq 0, j \in N\}$$

Then go to Step 1.

Step 4:

If Z_k^* involves M_i or N, go to Step 5. Otherwise let k=k+1, $Z_0 = Z_k^*$ and go to Step 1.

Step 5:

If k=0 then the problem (11) is infeasible. Otherwise the obtained solution at the last iteration is the optimal solution.

4. Computational results

Two following examples are solved by use of the genetic algorithm proposed in this article to illustrate the feasibility and efficiency of the proposed algorithm. The first example is LQBP and the second example is LFBP.

Example 1

Consider the following linear quadratic bi-level programming problem [16].

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$$\max_x \quad x_1 + x_2 + 3y_1 - y_2$$

$$s.t \ \max_y \quad 5y_1 + 8y_2 + (x_1, x_2, y_1, y_2)\begin{pmatrix} 1 & 3 & 2 & 0 \\ 3 & 1 & 4 & -2 \\ 2 & 4 & -2 & 1 \\ 0 & -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

$$\begin{aligned} s.t \quad & x_1 + x_2 + y_1 + y_2 \le 12, \\ & -x_1 + x_2 \le 2, \\ & 3x_1 - 4y_2 \le 5, \\ & y_1 + y_2 \le 4, \\ & x_1, x_2, y_1, y_2 \ge 0. \end{aligned}$$

Using KKT conditions following problem is obtained:

$$\begin{aligned} \max_x \quad & x_1 + x_2 + 3y_1 - y_2 - M_1(4x_1 + 8x_2 - 4y_1 + 2y_2 - u_1 - u_4 + 5) \\ & - M_2(-4x_2 + 10y_2 - u_1 + 4u_3 - u_4 + 8) \\ & - M_3(u_1w_1 + u_2w_2 + u_3w_3 + u_4w_4) - M_4(v_1y_1 + v_2y_2) \\ s.t \quad & \\ & x_1 + x_2 + y_1 + y_2 \le 12, \\ & -x_1 + x_2 \le 2, \\ & 3x_1 - 4y_2 \le 5, \\ & y_1 + y_2 \le 4, \\ & x_1, x_2, y_1, y_2, v_1, v_2, u_i, w_i \ge 0 \ , i = 1,2,3,4. \end{aligned}$$

Step 1: the following problem is feasible because (0, 0, 0, 0) is a feasible solution.

$$\begin{aligned} & x_1 + x_2 + y_1 + y_2 \le 12, \\ & -x_1 + x_2 \le 2, \\ & 3x_1 - 4y_2 \le 5 \\ & y_1 + y_2 \le 4, \end{aligned}$$

Now we find a basic feasible solution:

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$$B = [a_5, a_6, a_7, a_8] = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 12 \\ 2 \\ 5 \\ 4 \end{bmatrix} \quad , \quad x_N = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the associated basic inverses as follows:

$$H^{-1} = B^{-1} = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

Step 2: By replacing the basic variables into (21), the objective function is obtained.

$$z_0^* = 0$$

Now we should calculate $z_j - c_j$,

$$z_1 - c_1 = c_B B^{-1} a_1 - c_1 = -1 \qquad z_2 - c_2 = c_B B^{-1} a_2 - c_2 = -1$$

for all non-basic variables:

$$z_3 - c_3 = c_B B^{-1} a_3 - c_3 = -3 \quad z_4 - c_4 = c_B B^{-1} a_4 - c_4 = +1$$

Where $c_B = (0,0,0,0)$.

Step 3: According to (15), x_3 has the smallest $z_j - c_j$ therefore x_3 should be entered into the basis. Also the leaving variable is u_4 because of the usual minimum ratio rule:

$$\min\{\frac{r_1}{y_{13}}, \frac{r_4}{y_{43}}\} = \min\{\frac{12}{1}, \frac{4}{1}\} = 4 = \frac{r_4}{y_{43}}.$$

Thus x_3 is substituted u_4Then go to Step 1.

Step 4: Because $z_0^* = 0$ then k=k+1, $z_0 = z_0^* = 0$ and go to Step 1.

Now the first main iteration has been finished, but the algorithm has not been finished yet. Therefore we continue this process until the optimal solution is obtained after seven main iterations according to the Table 1.

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It is easy to show that by relaxing the u's and v variables (by fixing them on zero or one) in the main problem, we can obtain upper bounds for the problem which might be not promising as expected. By enumeration of possible relaxation the best upper bound is shown in Table 1.

$$w_1 = w_2 = w_3 = w_4 = 0 \;,\; v_1 = v_2 = 0 \quad \Rightarrow \quad z^* = 23$$

Table 1 comparison the best solutions - Example 1

Best solution by our method		Optimal solution by references [16, 20]		The best relaxation upper bound	
$(x_1^*, x_2^*, y_1^*, y_2^*)$	z^*	$(x_1^*, x_2^*, y_1^*, y_2^*)$	z^*	$(x_1^*, x_2^*, y_1^*, y_2^*)$	z^*
$(\frac{19}{3}, \frac{5}{3}, 4, 0)$	19.9999	(6.3125, 1.6875, 4.0000, 0.0000)	20	(7.25, 2.23, 4.51, 0.00)	23

According to the Table 1, the best solution by the proposed algorithm equals to the optimal solution exactly. It can be seen that the proposed method is efficient and feasible from the results.

Example 2

The following problem is linear fractional bi-level programming problem [13].

$$
\begin{aligned}
\max_{x} \quad & \frac{5 - 2x - y}{2 + x + y} \\
s.t \ \max_{y} \quad & y \\
s.t \quad & -5x - 3y \le -15, \\
& -x + 4y \le 28, \\
& 2x + 3y \le 32, \\
& 2x + 2y \le 26, \\
& 2x - y \le 13, \\
& x - 4y \le 3, \\
& x, y \ge 0.
\end{aligned}
$$

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Appling KKT conditions the above problem convert to this problem:

$$
\begin{aligned}
\max_{x} \quad & \frac{5-2x-y}{2+x+y} \\
s.t \quad & -5x-3y+w_1=-15, \\
& -x+4y+w_2=28, \\
& 2x+3y+w_3=32, \\
& 2x+2y+w_4=26, \\
& 2x-y+w_5=13, \\
& x-4y+w_6=3, \\
& -3u_1+4u_2+3u_3+2u_4-u_5-4u_6-v=-3, \\
& w_1u_1+w_2u_2+w_3u_3+w_4u_4+w_5u_5+w_6u_6=0, \\
& yv=0, \\
& x,y,v,w_i,u_i \geq 0\ ,\ i=1,\ldots,6.
\end{aligned}
$$

By enumeration of possible relaxation, the best upper bound is

$$w_1=w_2=w_3=w_4=0,\ u_5=u_6=0\ ,\ v=0 \quad \Rightarrow\ z^*=1.95$$

In genetic algorithm the initial population is created according to the proposed rules in section 4. Also the best solution is produced by the following chromosome:

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Choosing $u_1=u_2=u_6=0\ ,\ w_3=w_4=w_5=0\ ,\ v=0,$ by the proposed genetic algorithm, the optimal solution is obtained. The best solution is $(x^*,y^*)=(8.66,4.33)$ and the upper level's objective function is 1.66 also the lower level's objective function is 8.66. The results are all close to the exact values in Ref [12, 21]. Behavior of the variables by modified simplex method has been show in figure 1.

5. Conclusion and future work

In this paper, we used the KKT conditions to convert the problem into a single level problem. Then we presented a genetic method and a modified simplex method for solving linear-quadratic bi-level programming and linear-fractional bi-level programming problems. Comparing with the results of previous methods, both algorithms have better numerical results and present better solutions in much less times. The best solutions produced by proposed algorithms are feasible unlike the previous best solutions by other researchers.

In the future works, the following should be researched:

(1) Examples in larger sizes can be supplied to illustrate the efficiency of the proposed hybrid algorithm.

(2) Research to use other unconstraint optimization methods such as Quasi – Newton for solving linear BLP.

(3) Showing the efficiency of the proposed algorithms for solving other kinds of BLP.

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